REVIEW ARTICLE

SAMRIDDHI Volume 13, Issue 2, 2021 Print ISSN: 2229-7111 Online ISSN: 2454-5767

Cooperative Phenomenon of Coupled Rössler Oscillators under the Simultaneous Presence of Diffusive and Environmental Coupling

Nirmalendu Hui and Arjun Das*

Department of Physics, Krishnagar Government College, Krishnagar, Nadia, West Bengal, India-741101.

Abstract

In the nonlinear science chaos synchronization and suppression is an important problem. Here, we discuss how synchronisation and cessation of oscillation is appeared in two coupled chaotic system depending on coupling parameter under simulataneous presence of direct and indirect coupling. We have performed linear stability analysis and derive the explicit conditions of getting amplitude death which agree well with numerical results.

Keywords: *Rössler* Oscillators, chaotic oscillators, homogeneous steady state, Chaos

SAMRIDDHI : A Journal of Physical Sciences, Engineering and Technology (2021); DOI: 10.18090/samriddhi.v13i02.17

Introduction

Nature is full of non-linear system that might be periodic or chaotic, delayed or non-delayed and they are rarely isolated. Therefore the study of collective behavior of two or more coupled system, named as cooperative phenomenon, is important in many fields like physical science, engineering, biological science *etc.*[1]. Depending on coupling scheme, feedback loop, coupling strength parameter, time delay etc. the interacting dynamical systems collectively emerges several important phenomena like synchronisation, amplitude death (AD) *etc.*[2]. In simple, when two or more identical or nonidentical systems maintain same dynamics (that may also be a function of one's dynamics) by adjusting some property like amplitude and phase then it is called synchronization. Pecora and Carrol [3] first showed that under certain conditions synchronization of two chaotic systems is possible. In last few decades after their work many research had been done on synchronization and explore various forms of synchronization [4-7].

Along with synchronization the oscillation quenching, loss of rhythmic activity, in coupled oscillators is also an important topic of research in many fields of natural sciences, such as physics, biology, engineering, [2,8,9] *etc*. The amplitude death (AD) state is a stable homogeneous steady state (HSS), arises in the coupled oscillators under some parametric conditions [2,10-12] when they collectively go to a stable fixed point due to their mutual interaction and/or interaction with environment. This is relevant when suppression of unwanted oscillations are to be required like laser system [13], multi-module floating airport [14], neuronal systems [15], electronic circuits [9,16] etc.

Corresponding Author: Arjun Das, Department of Physics, Krishnagar Government College, Krishnagar, Nadia, West Bengal, India-741101, Email: arjundas2005@gmail.com

How to cite this article: Hui, N., Das, A. (2021). Cooperative Phenomenon of Coupled Rössler Oscillators under the Simultaneous Presence of Diffusive and Environmental Coupling. *SAMRIDDHI : A Journal of Physical Sciences, Engineering and Technology,* 13(2), 168-172.

Source of support: Nil

Conflict of interest: None

Chaos is aperiodic long-term behavior in a system that exhibit sensitive dependence on initial conditions. Poincaré (in 1800s) first glimpsed the possibility of chaos, in which a system exhibit aperiodic behavior that depends on initial conditions.[17] Invention of high speed computer enables the scientists to experiment with the nonlinear equation and eventually in 1963 Lorenz discover the chaotic motion on a strange attractor. In a chaotic system there are infinite number of unstable periodic orbits embedded in the chaotic attractor. Chaotic systems are inherently unpredictable due to the extreme critical dependence on the initial conditions. This unpredictable nature has both beneficial effect, as it may be strengthen the secured communication, and detrimental effect as it may leads irregular operations, disaster and collapse etc. Thus proper control of chaotic motion, which includes both chaos synchronization and suppression, is very important.

Among various coupling scheme direct-indirect coupling scheme has its own importance as in many system particularly in biological system, oscillators not only interact diffusively but also interact indirectly through their common environment. In general direct-indirect coupling topology is used to achieve oscillation quenching [16,18]. Time delay, which is inubiqutous in natuer, facilitate the emergence of amplitude death (AD) and oscillation death (OD) [9,10,19-24]. In this study we investigate thecoupled dynamics of two coupled chaotic *Rössler* system using direct-indirect coupling scheme. We explained our study both theoretically and numerically.

The rest of the paper is organized as follows: we describe the general mathematical model of the coupled system consisting of direct-indirect coupling scheme with propagation delay in system description section. We explain the approximate stability analysis in stability analysis section and observation from numerical analysis for the coupled system later. Next we summarize the main findings of the whole study in conclusion.

System Description: Chaotic Oscillator without Delay; Rössler Oscillator

We consider two identical Rössler oscillators which are in chaotic mode coupled directly through diffusive coupling with coupling strength d as well as indirectly through a common environment s coupling strength ε. The mathematical model looks like equation (1)

$$
\begin{aligned}
u_1 &= -v_1 - w_1 + d\left(u_2 - u_1\right) + \varepsilon s \\
v_1 &= u_1 + a v_1 \\
w_1 &= b + w_1\left(u_1 - c\right) \\
u_2 &= -v_2 - w_2 + d\left(u_1 - u_2\right) + \varepsilon s \qquad (1) \\
v_2 &= u_2 + a v_2 \\
w_2 &= b + w_2\left(u_2 - c\right) \\
w_2 &= b + w_2\left(u_2 - c\right)
\end{aligned}
$$

Where a, b and c are controlling parameters of *Rössler* system. Depending on the values of a, b and c the system behave as either periodic or chaotic. For lower value of c it show periodic nature but with increase of c the system becomes chaotic through period doubling route. The bifurcation diagram in Figure 1(b) depicts it clearly. For the value of $a = 0.2$, $b = 0.2$ and $c = 5$ the uncoupled oscillators show their chaotic nature which is shown by phase diagram (Figure 1(a)) and time series plot (Figure.2(a)). d and ε determine the strength of the diffusive (direct) coupling and indirect coupling with environment. An over-damped oscillator with damping coefficient $\kappa(<0)$ is used to model environment s and it decays monotonically toward the zero steady state in absence of both oscillator and remains in that dormant state.

Stability analysis

To get the steady state condition we have performed stability analysis. For this, linearizing equation (1) around the equilibrium point $x^* \equiv \left(u_i^*, v_i^*, w_i^*, z^*\right)^T$ by setting $x(t) = x^* + \delta x(t)$ where $u_i^* = \frac{1}{2}\left(c \pm \sqrt{c^2 - 4\frac{a\kappa b}{\kappa - \varepsilon^2 a}}\right)$, $v_i^* = -u_i^* / a$, $w_i^* = \frac{\kappa - \varepsilon^2 a}{a\kappa} u_i^*$, $z^* = -\varepsilon u_i^* / \kappa$ for $i = 1,2$ we obtain the following characteristic equation with $\alpha = u_i^* - c$ and $w_i^* = \beta$.

$$
\begin{vmatrix}
-d-\lambda & -1 & -1 & d & 0 & 0 & \varepsilon \\
1 & a-\lambda & 0 & 0 & 0 & 0 & 0 \\
\beta & 0 & \alpha-\lambda & 0 & 0 & 0 & 0 \\
d & 0 & 0 & -d-\lambda & -1 & -1 & \varepsilon \\
0 & 0 & 0 & 1 & a-\lambda & 0 & 0 \\
0 & 0 & 0 & \beta & 0 & \alpha-\lambda & 0 \\
\frac{-1}{2}\varepsilon & 0 & 0 & \frac{-1}{2}\varepsilon & 0 & 0 & -\kappa-\lambda
\end{vmatrix} = 0 \qquad (2)
$$

Now let left hand side of equation (2) is equal to I. Then after simplify the determinant (I) one can get

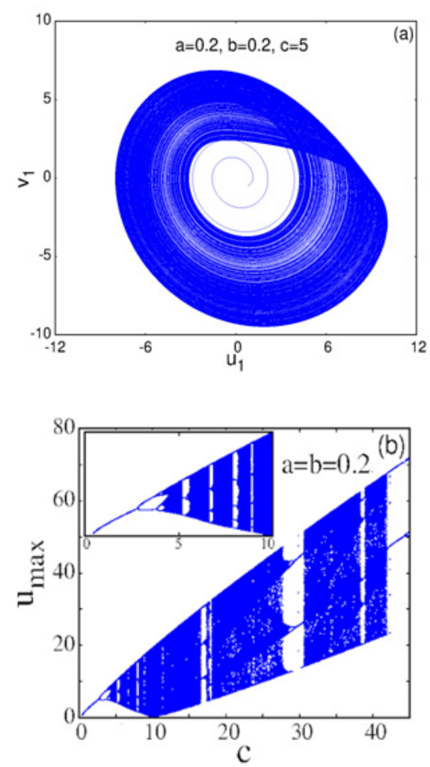

Figure 1: (a) Phase diagram of the chaotic uncoupled *Rössler* system with $a = 0.2$, $b = 0.2$, $c = 5$. **(b)** Bifurcation diagram for Rssler system for varying parameter c (inset figure shows the zoom of the plot in the range $c = 0$ to $c = 10$). With increasing c system enters into chaotic zone through multiple periode doubling routes

$$I = (-d-\lambda)(a-\lambda)(\alpha-\lambda)I_1 + (\alpha-\lambda)I_1 - I_2 - dI_3 + \varepsilon I_4 \tag{3}$$

Where

$$I_1 = \begin{vmatrix} -d-\lambda & -1 & -1 & \varepsilon \\ 1 & a-\lambda & 0 & 0 \\ \beta & 0 & \alpha-\lambda & 0 \\ \frac{-1}{2}\varepsilon & 0 & 0 & -\kappa-\lambda \end{vmatrix}$$

$$= \frac{1}{2}\varepsilon^2(\alpha-\lambda)(a-\lambda) - \beta(\kappa+\lambda)(a-\lambda) - (\kappa+\lambda)(\alpha-\lambda)\left(1-(d+\lambda)(a-\lambda)\right) \tag{4}$$

$$I_2 = \begin{vmatrix} 1 & a-\lambda & 0 & 0 & 0 & 0 \\ \beta & 0 & 0 & 0 & 0 & 0 \\ d & 0 & -d-\lambda & -1 & -1 & \varepsilon \\ 0 & 0 & 1 & a-\lambda & 0 & 0 \\ 0 & 0 & \beta & 0 & \alpha-\lambda & 0 \\ \frac{-1}{2}\varepsilon & 0 & \frac{-1}{2}\varepsilon & 0 & 0 & -\kappa-\lambda \end{vmatrix} \tag{5}$$

$$= -\beta(a-\lambda)I_1$$

$$I_3 = \begin{vmatrix} 1 & a-\lambda & 0 & 0 & 0 & 0 \\ \beta & 0 & \alpha-\lambda & 0 & 0 & 0 \\ d & 0 & 0 & -1 & -1 & \varepsilon \\ 0 & 0 & 0 & a-\lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha-\lambda & 0 \\ \frac{-1}{2}\varepsilon & 0 & 0 & 0 & 0 & -\kappa-\lambda \end{vmatrix} \tag{6}$$

$$= (a-\lambda)^2(\alpha-\lambda)^2\left(-\frac{1}{2}\varepsilon^2 - d(\kappa+\lambda)\right)$$

$$I_4 = \begin{vmatrix} 1 & a-\lambda & 0 & 0 & 0 & 0 \\ \beta & 0 & \alpha-\lambda & 0 & 0 & 0 \\ d & 0 & 0 & -d-\lambda & -1 & -1 \\ 0 & 0 & 0 & 1 & a-\lambda & 0 \\ 0 & 0 & 0 & \beta & 0 & \alpha-\lambda \\ \frac{-1}{2}\varepsilon & 0 & 0 & \frac{-1}{2}\varepsilon & 0 & 0 \end{vmatrix} \tag{7}$$

$$= -(a-\lambda)^2(\alpha-\lambda)^2 d\varepsilon/2 - (a-\lambda)^2(\alpha-\lambda)^2(d+\lambda)\varepsilon/2 + (a-\lambda)(\alpha-\lambda)^2\varepsilon/2 + (a-\lambda)^2(\alpha-\lambda)\beta\varepsilon/2$$

Now putting the value of I_1, I_2, I_3, I_4 in equation (3) and further simplification gives

$$\begin{aligned} I &= (\kappa+\lambda)A^2 - \varepsilon^2(\alpha-\lambda)(a-\lambda)A + (\alpha-\lambda)^2(a-\lambda)^2 d[\varepsilon^2 - d(\kappa+\lambda)] \\ &= [(\kappa+\lambda)A - (\alpha-\lambda)(a-\lambda)(\varepsilon^2 - d(\kappa+\lambda))][A - d(\alpha-\lambda)(a-\lambda)] \end{aligned} \tag{8}$$

With

$$A = \beta(a-\lambda) + (\alpha-\lambda) - (a-\lambda)(\alpha-\lambda)(d+\lambda) \tag{9}$$

So after all those above simplification the equation (3) looks like

$$[(\kappa+\lambda)A - (\alpha-\lambda)(a-\lambda)(\varepsilon^2 - d(\kappa+\lambda))][A - d(\alpha-\lambda)(a-\lambda)] = 0 \tag{10}$$

The steady state of the coupled systems are locally stable if and only if all the roots of the characteristic equations (2) are in the left half complex plane that is real part of eigen value is negative. To get the stability condition of the fixed points we apply the Routh–Hurwitz stability criterion. If real part of maximum eigen value is negative then we get Amplitude death (AD) region. And we notice that this region depends on parameter value. From equation (10) we get either

$$(\kappa+\lambda)A - (a-\lambda)(\alpha-\lambda)\left(\varepsilon^2 - d(\kappa+\lambda)\right) = 0 \tag{11}$$

Or, $A - d(a-\lambda)(\alpha-\lambda) = 0$ (12)

Now if we simplify more, the equation (11) looks like

$$A_0\lambda^4 + A_1\lambda^3 + A_2\lambda^2 + A_3\lambda^1 + A_4\lambda^0 = 0 \tag{13}$$

Where

$$\begin{gathered} A_0 = 1 \\ A_1 = \kappa - (a+\alpha) \\ A_2 = 1 + \beta + \varepsilon^2 + a\alpha - \kappa(a+\alpha) \\ A_3 = a\kappa\alpha - \varepsilon^2(a+\alpha) - (\alpha-\kappa) - \beta(a-\kappa) \\ A_4 = a\alpha\varepsilon^2 - \alpha\kappa - a\beta\kappa \end{gathered} \tag{14}$$

According to the Routh-Hurwitz stability criterion A_0, A_1, B_1, C_1, A_4 (values are given in equations (14) and (15) must be +ve for stable equilibrium. Where

$$\begin{aligned} B_1 &= \frac{(A_1A_2 - A_3A_0)}{A_1} \\ B_2 &= A_4 \\ C_1 &= \frac{(B_1A_3 - B_2A_1)}{B_1} \end{aligned} \tag{15}$$

Again simplifying the equation (12) we get the following inequalities as equation (16) for stable fixed point:

$$\begin{aligned} & 2a\alpha d - \alpha\beta - \alpha > 0 \\ & 2d - (a+\alpha) > 0 \\ & [2d - (a+\alpha)][1 + \beta - 2d(a+\alpha) + a\alpha] > (2a\alpha d - \alpha\beta - \alpha) \end{aligned} \tag{16}$$

To get AD we must have $A_0 \geq 0$, $A_1 \geq 0$, $B_1 \geq 0$, $C_1 \geq 0$, $A_4 \geq 0$ along with the inequalities given by equation (16). So if any one of those conditions does not obey then system will not be stable and become oscillatory.

Numerical Results

We use fourth-order *Runge–Kutta* algorithm with step size $h = 0.001$ to integrate equation (1) numerically. We fixed the system parameters $a = 0.2$, $b = 0.2$ and $c = 5$ so that individual oscillator system shows chaotic oscillations. Here, We consider the system is in death state when the amplitude of oscillation is either zero or less than equal to 0.001 after running the time interval 0 to 4000.

Presence of only diffusive coupling:

First we consider only diffusive coupling by setting $\varepsilon = 0$ i.e, there is no effect of environment. As we increase the value of diffusive coupling strength d we notice that coupled chaotic systems become synchronised to each other. We calculate synchronisation error (e_s) to realize and represent the synchronisation where $e_s = u_1 - u_2$ at a certain time. Zero value of synchronisation error (e_s) indicates total synchronisation. The numerical results are represented in Figure 2 which clearly shows that beyond a certain value of d one can get synchronisation. Figure 2(a) represents the time series of non-coupled chaotic systems which are not synchronised and Figure 2(b) also confirms that which shows the plot of synchronisation error (e_s) with time. But as we increase the couling strength both systems gradually becomes synchronised with each other beyond a certain value of d (Figure 2(c) with $d = 0.05$) and one can easily realize it as after acertain time corresponding e_s becomes zero (Figure 2(d)).

Dynamics in $d-\varepsilon$ space:

Next we consider the indirect coupling through their common environment *i.e.,* $\varepsilon \neq 0$. We fix the value $\kappa = 2$

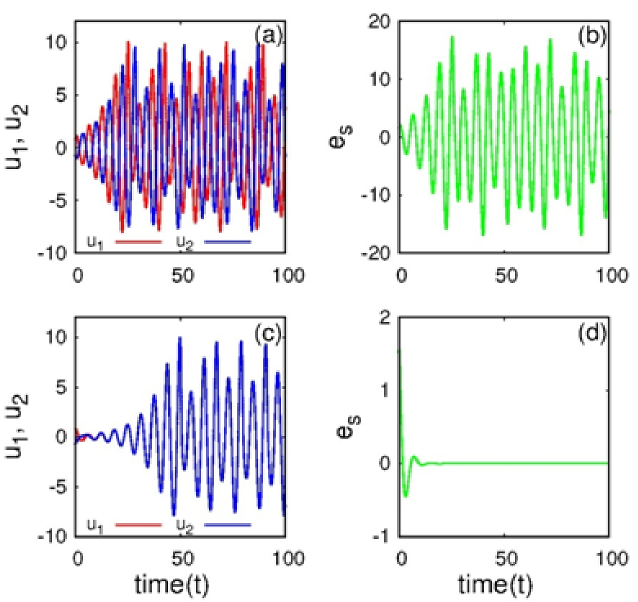

Figure 2: (a) Time series of u_1 (red) and u_2 (blue) in absence of any coupling i.e, $d = 0$ and $\varepsilon = 0$ showing non-coupled oscillators are in chaotic mode and as well as they are in non synchronous states. **(b)** Plot of synchronisation error *vs* time for $d = 0$ and $\varepsilon = 0$. **(c)** Time series of u_1 (red) and u_2 (blue) in presence of only diffusive coupling. Here $d = 0.05$ and $\varepsilon = 0$ showing synchronisation of coupled oscillators. **(d)** plot of synchronisation error *vs* time for $d = 0.05$ and $\varepsilon = 0$. Here $a = 0.2$, $b = 0.2$ and $c = 5$.

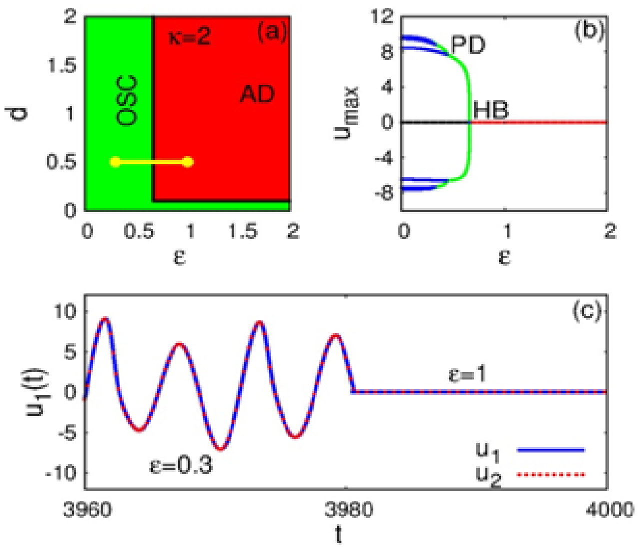

Figure 3: (a) Two-parameter stability diagram in the $d-\varepsilon$ space ($\kappa = 2$) showing AD zone (Red/dark grey) and oscillatory zone (Green/light grey). Black line indicates theoretical border line. **(b)** Bifurcation diagram with changing ε along the dashed yellow line of **(a)** ($d = 0.5$). **(c)** Time series plot showing oscillation at ($\varepsilon = 0.3$) and AD ($\varepsilon = 1$) with. Points on the dashed yellow line indicate the parameters for which time series are generated. Here $a = 0.2$, $b = 0.2$ and $c = 5$.

and we investigate the coupled dynamics in $d-\varepsilon$ region. The two-parameter stability diagram in the $d-\varepsilon$ space ($\kappa = 2$ is shown in Figure 3(a) showing AD zone (Red/dark grey) and oscillatory zone (Green/light grey). Solid black line indicates theoretical border line originating from stability analysis condition. With the increase of effect of environment coupling that is with the increase of ε value the coupled system enter into amplitude death (AD) zone through inverse hopf-bifurcation. For detail analysis we draw the bifurcation diagram (Figure 3(b)) with changing ε which indicates hopf point exists at $\varepsilon = 0.664$ denoted as HB. In that figure 'PD' denotes where period doubling occurs. Blue points indicate unstable limit cycle. We also draw time series plot for those two points indicated by yellow color in Figure 3(a). This time-series plot clearly shows the as we increase the ε value the coupled system transit from oscillatory zone (here $\varepsilon = 0.3$) to AD zone (here, $\varepsilon = 1$).

Conclusion

In this paper we have studied coupled dynamics of chaotic *Rössler* oscillator using direct and indirect coupling scheme. We explore how diffusive and environment coupling affects the collective dynamics of two coupled non-delayed chaotic oscillators. We systematically represent our results getting from numerical simulation which may help to realize the couple dynamics. In absence of indirect coupling, diffusive coupling strengthen the synchronisation effect. But indirect coupling through environment along with diffusive coupling helps to transit the coupled systems from oscillatory zone to amplitude death zone. We have performed theoretical stability analysis which solidify those numerical findings. We believe that our study is significant in the context of emergence of AD and mechanism behind transition from chaotic oscilllations to AD.

Acknowledgement

We are thankful to Department of Physics, Krishnagar Government College, West Bengal, India for providing necessary research facilities. We acknowledge Dr. Tanmoy Banerjee, Professor, The University of Burdwan for his valuable discussions and to sort out the problem.

References

[1] Pikovsky, A. S., Rosenblum, M. G., & Kurths J. (2001). Synchronization. A Universal Concept in Nonlinear Sciences, *Cambridge University Press*, Cambridge, UK.

[2] Saxena, G., Prasad, A., & Ramaswamy, R. (2012). Amplitude death: The emergence of stationarity in coupled nonlinear systems, *Phys. Rep.*, 521, 205.

[3] Pecora, L.M., & Carroll, T.L. (1990). Synchronization in chaotic systems, *Phys. Rev. Lett.*, 64, 821.

[4] Rulkov, N.F., Sushchik, M.M., Tsimring, L.S., & Abarbanel, H.D.I. (1995). Generalized synchronization of chaos in directionally coupled chaotic systems, *Phys. Rev. E*, 51, 980.

[5] Rosenblum, M.G., Pikovsky, A.S., & Kurths, J. (1996). Phase Synchronization of Chaotic Oscillators, *Phys. Rev. Lett.*, 76, 1804.

[6] Rosa, E.R., Ott, E., & Hess, M.H. (1998). Transition to Phase Synchronization of Chaos, *Phys. Rev. Lett.*, 80, 1642.

[7] Boccaletti, S., Kurths, J., Osipov, G., Valladares, D.L., & Zhou, C.S. (2002). The synchronization of chaotic systems, *Phys. Rep.*, 366, 1-101.

[8] Koseska, A., Volkov, E., & Kurths, J. (2013). Oscillation quenching mechanisms: Amplitude vs. oscillation death, *Phys. Rep.*, 531, 173.

[9] Banerjee, T., & Biswas, D. (2013). Amplitude death and synchronized states in nonlinear time-delay systems coupled through mean-field diffusion, *Chaos,* 23, 043101.

[10] Reddy, D. V. R., Sen, A., & Johnston, G. L. (1998). Time Delay Induced Death in Coupled Limit Cycle Oscillators, *Phy. Rev. Lett*, 80, 5109.

[11] Reddy, D. V. R., Sen, A., & Johnston, G. L. (2000). Experimental Evidence of Time-Delay-Induced Death in Coupled Limit-Cycle Oscillators, *Phy. Rev. Lett,* 85, 3381.

[12] Prasad, A. (2005). Amplitude death in coupled chaotic oscillators, *Phy. Rev. E*, 72, 056204.

[13] Kumar, P., Prasad, A., & Ghosh, R. (2008). Stable phase-locking of an external-cavity diode laser subjected to external optical injection, *J. Phys. B*, 41, 135402.

[14] Zhang, H., Xu, D., Lu, C., Qi, E., Hu, J., & Wu, Y. (2015). Amplitude death of a multi-module floating airport, *Nonlinear Dynamics*, 79, 2385–2394.

[15] Ermentrout, G. B., & Kopell, N. (1990). Oscillator Death in Systems of Coupled Neural Oscillators, *SIAM J. Appl. Math.*, 50, 125.

[16] Ghosh, D., & Banerjee, T. (2014). Transitions among the diverse oscillation quenching states induced by the interplay of direct and indirect coupling, Phy. Rev. E, 90, 062908.

[17] Strogatz, S. H. (2012). Sync: How Order Emerges from Chaos In the Universe, *Nature, and Daily Life,* Hyperion, New York.

[18] Resmi, V., Ambika, G., & Amritkar, R.E. (2011). General mechanism for amplitude death in coupled systems, *Phys. Rev. E*, 84, 046212(1–12).

[19] Atay, F. M. (2003). Distributed Delays Facilitate Amplitude Death of Coupled Oscillators, *Phy. Rev. Lett,* 91, 094101.

[20] Zou, W., & Zhan, M. (2009). Partial time-delay coupling enlarges death island of coupled oscillators, *Phy. Rev. E*, 80, 065204.

[21] G. Saxena, A. Prasad, R. Ramaswamy, Dynamical effects of integrative time-delay coupling, *Phy. Rev. E*, 82, 017201, 2010.

[22] Banerjee, T., & Biswas, D. (2013). Synchronization in hyperchaotic time-delayed electronic oscillators coupled indirectly via a common environment, *Nonlinear Dynamics*, 73, 2024–2048.

[23] Gjurchinovski, A., Zakharova, A., & Scholl, E. (2014). Amplitude death in oscillator networks with variable-delay coupling, *Phy. Rev. E*, 89, 032915.

[24] Biswas, D., Hui, N., & Banerjee, T. (2017). Amplitude death in intrinsic time-delayed chaotic oscillators with direct-indirect coupling: The existence of death islands, *Nonlinear Dyn.*, 88, 2783-2793.